Towards Berglund-Hübsch mirror symmetry

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with B. Gammage

Matrices $\rightsquigarrow$ Polynomials

$A = (a_{ij})$ $n \times n$ $\mathbb{Z}_{\geq 0}$ matrix

$$W = \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{a_{ij}} \in \mathbb{C}[x_1, \dots, x_n]$$

Berglund-Hübsch transpose

$$\check{W} = \sum_{i=1}^{n} \prod_{j=1}^{n} \check{x}_j^{a_{ji}} \in \mathbb{C}[\check{x}_1, \dots, \check{x}_n]$$

Proposal (Berglund–Hübsch, '92)

The LG models $(\mathbb{C}^n, w)$ and $(\mathbb{C}^n, \check{w})$

are mirror if the polynomials are invertible

sensitive to the geometry of the singularities of $w, \check{w}$

- both have isolated singularities at 0
- $w, \check{w}$ are quasi-homogeneous
- $\det A \neq 0$

Examples

① Fermat / Brieskorn-Pham

$$A = \begin{pmatrix} p_1 & & \\ & \ddots & \\ & & p_n \end{pmatrix}$$

self-transpose

$$W = x_1^{p_1} + \dots + x_n^{p_n}$$

$$\check{W} = \check{x}_1^{p_1} + \dots + \check{x}_n^{p_n}$$

Quasi-homogeneous with weight of $x_i, \check{x}_i \propto \frac{1}{p_i}$

Invertible $\iff$ $p_i \geq 2 \quad \forall i$

Special cases

- A_k $\qquad x_1^{k+1} + x_2^2 + \dots + x_n^2$
- E_6 $\qquad x_1^4 + x_2^3 + x_3^2 + \dots + x_n^2$
- E_8 $\qquad x_1^5 + x_2^3 + x_3^2 + \dots + x_n^2$

② Chain

$$A = \begin{pmatrix} p_1 & 1 & & \\ & p_2 & 1 & \\ & & \ddots & 1 \\ & & & p_n \end{pmatrix}$$

$$W = x_1^{p_1} x_2 + x_2^{p_2} x_3 + \dots + x_{n-1}^{p_{n-1}} x_n + x_n^{p_n}$$

$$\check{W} = \check{x}_1^{p_1} + \check{x}_1 \check{x}_2^{p_2} + \dots + \check{x}_{n-2} \check{x}_{n-1}^{p_{n-1}} + \check{x}_{n-1} \check{x}_n^{p_n}$$

Special cases

- D_k $\quad x_1^2 x_2 + x_2^{k-1} \quad (+ x_3^2 + \dots + x_n^2)$
- E_7 $\quad x_1^3 x_2 + x_2^3 \quad (+ x_3^2 + \dots + x_n^2)$

③ Loop

$$A = \begin{pmatrix} p_1 & 1 & & & \\ & p_2 & 1 & & \\ & & \ddots & & \\ & & & & 1 \\ 1 & & & & p_n \end{pmatrix}$$

$$W = x_1^{p_1} x_2 + x_2^{p_2} x_3 + \cdots + x_{n-1}^{p_{n-1}} x_n + x_n^{p_n} x_1$$

$$\check{W} = \check{x}_n \check{x}_1^{p_1} + \check{x}_1 \check{x}_2^{p_2} + \cdots + \check{x}_{n-2} \check{x}_{n-1}^{p_{n-1}} + \check{x}_{n-1} \check{x}_n^{p_n}$$

<u>Thm</u> (Kreuzer-Skarke) Every invertible polynomial is a Thom-Sebastiani sum of BP, chain, and loop polynomials.

More examples

strange duality

Tables from Ebeling-Takahashi "Strange duality of weighted homogeneous polynomials" Compositio 2011

TABLE 12. Arnold's strange duality.

Name	$\alpha_1, \alpha_2, \alpha_3$	f	f^t	$\gamma_1, \gamma_2, \gamma_3$	Dual
E_{12}	2, 3, 7	$x^2+y^3+z^7$	$x^2+y^3+z^7$	2, 3, 7	E_{12}
E_{13}	2, 4, 5	$x^2+y^3+yz^5$	$x^2+zy^3+z^5$	2, 3, 8	Z_{11}
E_{14}	3, 3, 4	$x^3+y^2+yz^4$	$x^3+zy^2+z^4$	2, 3, 9	Q_{10}
Z_{12}	2, 4, 6	$x^2+zy^3+yz^4$	$x^2+zy^3+yz^4$	2, 4, 6	Z_{12}
Z_{13}	3, 3, 5	$x^2+xy^3+yz^3$	$x^2y+y^3z+z^3$	2, 4, 7	Q_{11}
Q_{12}	3, 3, 6	$x^3+zy^2+yz^3$	$x^3+zy^2+yz^3$	3, 3, 6	Q_{12}
W_{12}	2, 5, 5	$x^5+y^2+yz^2$	$x^5+y^2z+z^2$	2, 5, 5	W_{12}
W_{13}	3, 4, 4	$x^2+xy^2+yz^4$	$x^2y+y^2z+z^4$	2, 5, 6	S_{11}
S_{12}	3, 4, 5	$x^3y+y^2z+z^2x$	$zx^3+xy^2+yz^2$	3, 4, 5	S_{12}
U_{12}	4, 4, 4	$x^4+zy^2+yz^2$	$x^4+zy^2+yz^2$	4, 4, 4	U_{12}

TABLE 13. Strange duality of the exceptional bimodal singularities.

Name	$\alpha_1, \alpha_2, \alpha_3$	f	f^t	$\gamma_1, \gamma_2, \gamma_3$	Dual
E_{18}	3, 3, 5	$x^3+y^2+yz^5$	$x^3+y^2z+z^5$	2, 3, 12	Q_{12}
E_{19}	2, 4, 7	$x^2+y^3+yz^7$	$x^2+y^3z+z^7$	2, 3, 12	$Z_{1,0}$
E_{20}	2, 3, 11	$x^2+y^3+z^{11}$	$x^2+y^3+z^{11}$	2, 3, 11	E_{20}
Z_{17}	3, 3, 7	$x^2+xy^4+yz^3$	$x^2y+y^4z+z^3$	2, 4, 10	$Q_{2,0}$
Z_{18}	2, 4, 10	$x^2+zy^3+yz^6$	$x^2+zy^3+yz^6$	2, 4, 10	Z_{18}
Z_{19}	2, 3, 16	$x^2+y^9+yz^3$	$x^2+y^9z+z^3$	2, 4, 9	E_{25}
Q_{16}	3, 3, 9	$x^3+zy^2+yz^4$	$x^3+zy^2+yz^4$	3, 3, 9	Q_{16}
Q_{17}	2, 4, 13	$x^3+xy^5+yz^2$	$x^3y+y^5z+z^2$	3, 3, 9	$Z_{2,0}$
Q_{18}	2, 3, 21	$x^3+y^8+yz^2$	$x^3+y^8z+z^2$	3, 3, 8	E_{30}
W_{17}	3, 5, 5	$x^2+xy^2+yz^5$	$x^2y+y^2z+z^5$	2, 6, 8	$S_{1,0}$
W_{18}	2, 7, 7	$x^7+y^2+yz^2$	$x^7+y^2z+z^2$	2, 7, 7	W_{18}
S_{16}	3, 5, 7	$x^4y+y^2z+z^2x$	$zx^4+xy^2+yz^2$	3, 5, 7	S_{16}
S_{17}	2, 7, 10	$x^6+xy^2+yz^2$	$x^6y+y^2z+z^2$	3, 6, 6	$X_{2,0}$
U_{16}	5, 5, 5	$x^5+zy^2+yz^2$	$x^5+zy^2+yz^2$	5, 5, 5	U_{16}

Homological B-H MS conjecture

(Takahashi, Futaki-Ueda, Lekili-Ueda, ...)

For transpose invertible polynomials w, $\check{w}$ there is a quasi-equivalence (of $\mathbb{Z}$-graded A_∞-categories)

$$mf(\mathbb{C}^n, \Gamma_w, w) \simeq \mathcal{F}(\check{w})$$

Γ_w-equivariant matrix factorisations of w

Fukaya-Seidel category of $\check{w} : \mathbb{C}^n \to \mathbb{C}$

$mf(\mathbb{C}^n, \Gamma_w, w)$

Γ_w = maximal diagonal symmetry group of w

$$= \{(t_1,\dots,t_{n+1}) \in (\mathbb{C}^*)^{n+1} : w(t_1x_1,\dots,t_nx_n) = t_{n+1}w(x_1,\dots,x_n)\}$$

finite extension of $\mathbb{C}^*$ via projection to t_{n+1}

Acts on $\mathbb{C}^n$ by forgetting t_{n+1}

Action preserves w up to rescaling so can consider Γ_w-equivariant matrix factorisations of w on $\mathbb{C}^n$

$\underline{mf(\mathbb{C}^n, \Gamma_w, w)}$

$$\cdots \longrightarrow E^0 \xrightarrow{\delta} E^1 \xrightarrow{\delta} E^2 \longrightarrow \cdots$$

- E^i Γ_w-equivariant vector bundle on $\mathbb{C}^n$
- $E^{i+2} = E^i \otimes \chi_w$
- $\delta^2 = \mathrm{id}_{E^i} \otimes w$

Non-equivariant

$$E^0 \underset{\delta}{\overset{\delta}{\rightleftarrows}} E^1 \qquad \delta^2 = w$$

χ_w = character $(t_1, \dots, t_{n+1}) \longmapsto t_{n+1}$

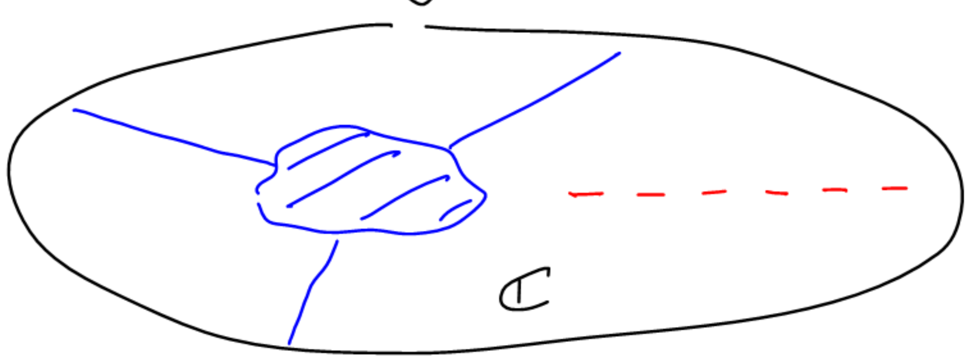
$\mathcal{F}(\check{w})$ à la Seidel

Objects Exact Lagrangians $L \subset \mathbb{C}^n$ such that

$\check{w}(L) =$ compact $\cup$ rays in $\mathbb{C}$ ~~^{n}~~ $\setminus \mathbb{R}_{>0}$

$$\mathbb{C}^n \xrightarrow{\check{w}}$$

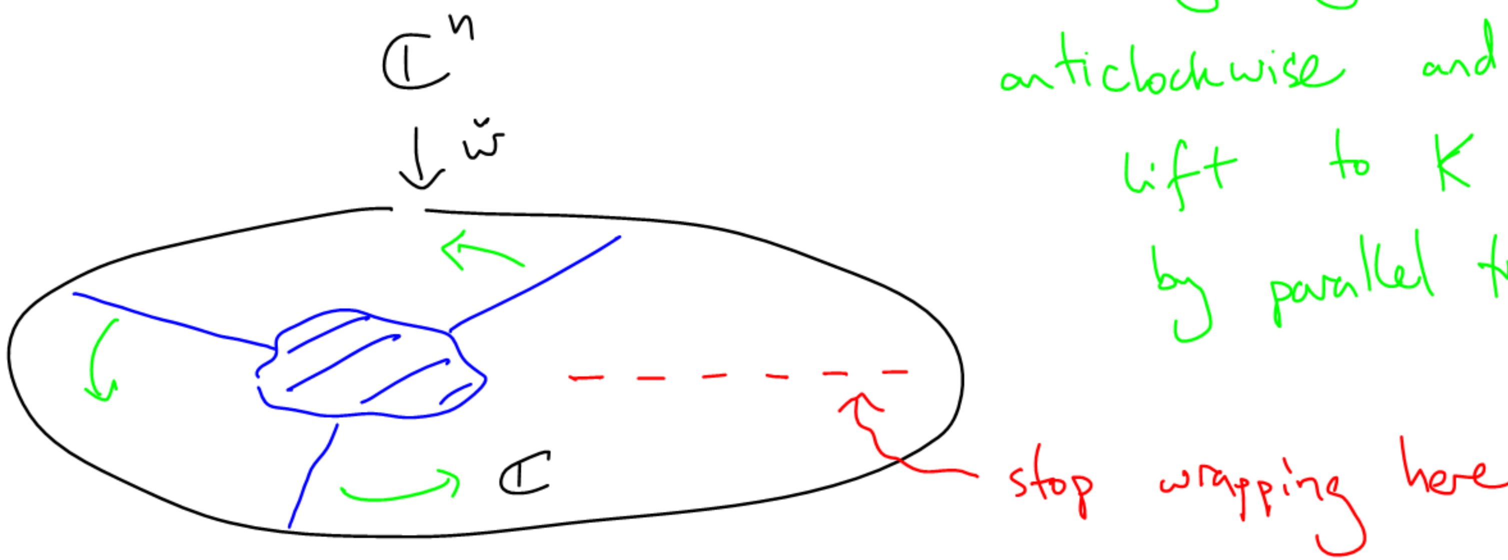
L must be fibrewise compact



$\mathcal{F}(\check{w})$ à la Seidel

Morphisms $\quad \hom^*(K, L) = CF^*(K^{wr}, L)$

wrap rays of $\check{w}(K)$ anticlockwise and lift to K by parallel transport



Known cases of $mf(\mathbb{C}^n, \Gamma_w, w) \simeq \mathcal{F}(\check{w})$

- Futaki–Ueda ('11): $\check{w}$ is type A or D, or Thom–Sebastiani sums of these (n arbitrary)

- Habermann–S ('19): $n = 2$ ($w, \check{w}$ arbitrary)

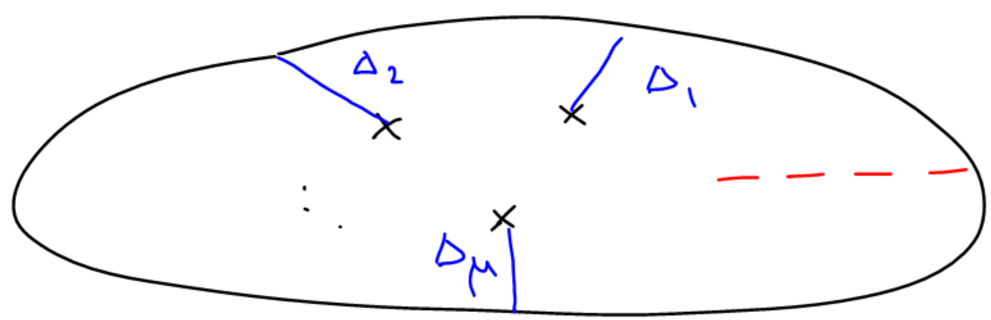
Proofs are indirect: show

$$mf(\mathbb{C}^n, \Gamma_w, w) \simeq \mathbb{C}Q\text{-mod} \simeq \mathcal{F}(\check{w})$$

quiver with relations

Traditional proof strategy

- Morsify $\check{w}$ by adding a small perturbation to obtain a Lefschetz fibration



- Seidel: Lefschetz thimbles give full exceptional collection $\langle \Delta_1, \ldots, \Delta_\mu \rangle$

  and can describe morphisms in terms of CF^* of vanishing cycles in Milnor fibre

- Find full exceptional collection in MF and show its morphisms are described by the same quiver

Disadvantages

so need to choose the right ones in order to match them up

- There's no canonical full exceptional collection in either category
- Choosing a Morsification of $\check{w}$ breaks its symmetries
- Gives no direct functor $\mathcal{F} \to MF$ and the indirect equivalence is not canonical

New approach – summary

- $mf(\mathbb{C}^n, \Gamma_w, w)$ does have canonical generators

the mx fac corresponding to skyscraper at 0

$\mathcal{O}_0$ and its Γ_w-twists

- Find mirror Lagrangians $\mathbb{L}$ in $\mathcal{F}$ — these carry an action of $G = \hat{\Gamma}_w$

character group

- Construct associated localised mirror functor

family Yoneda map (Cho–Hong–Lau) (inspired by Seidel, Sheridan)

$$\mathcal{F}(\check{w}) \longrightarrow mf(\mathbb{C}^n, \Gamma_w, w)$$

arises from G-action on $\mathbb{L}$

- Show it's a quasi-equivalence

Recent related work ('18 – '20)

B-side

Hirano – Ouchi: Construct strong full exceptional collection in mf for chain polys and describe endomorphisms as explicit quiver with relations

Aramaki – Takahashi: Construct a different full exceptional collection for chain polys, inspired by Orlik – Randell, and compute Euler pairing on K_0

Recent related work ('18 - '20)

B-side

Kravets — Constructs strong full exceptional collections in mf for all invertible polys with $n \leq 3$, and computes quivers

Favero - Kaplan - Kelly — Prove existence of full exceptional collections, of expected size, for all invertible polys

Recent related work ('18 – '20)

A-side

Varolgunes   Computes monodromy around chain singularity as conjectured by Orlik – Randell

Combined with Aramaki – Takahashi this proves HB-HMS for chain polynomials at the level of K_0 + Euler pairing

Recent related work ('18 – '20)

HMS

Lekili–Ueda Conjecture for mirror to Milnor fibre of $\check{w}$

$$mf(\mathbb{C}^{n+1}, \Gamma_w, w + x_0 x_1 \cdots x_n) \simeq \mathcal{W}(\check{w}^{-1}(1))$$

proved in various cases, including ADE and

$$\check{w} = \check{x}_1^{n+1} + \cdots + \check{x}_n^{n+1}$$

Habermann Proves Lekili–Ueda conjecture for compact Fukaya category $\mathcal{F}(\check{w}^{-1}(1))$ for $n = 2$

Key ingredients in new approach

- Description of the Lagrangians $\mathbb{L}$
- Localised mirror functor and its new graded version

Mirror symmetry heuristic

$$(\mathbb{C}^*)^n$$

partially compactify

$$\mathbb{C}^n$$

quotient by deck group

$$\mathbb{C}^n/\bar{G}$$

turn on potential

$$\left(\mathbb{C}^n/\bar{G},\ w(x_1,\dots,x_n)\right)$$

$$(\mathbb{C}^*)^n$$

turn on potential

$$\left((\mathbb{C}^*)^n,\ \check{y}_1+\dots+\check{y}_n\right)$$

pass to cover

$$\left((\mathbb{C}^*)^n,\ \check{w}(\check{x}_1,\dots,\check{x}_n)\right)$$

partially compactify

$$\left(\mathbb{C}^n,\ \check{w}(\check{x}_1,\dots,\check{x}_n)\right)$$

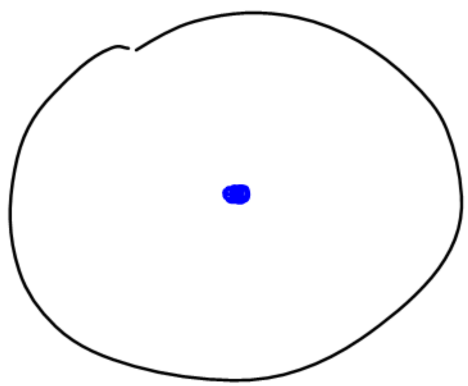
Mirror symmetry heuristic

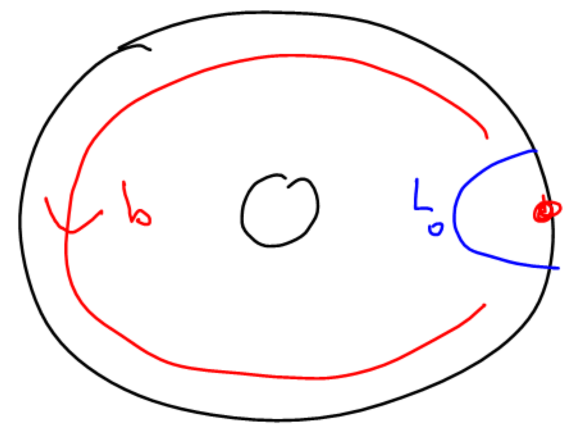
$$\mathcal{O}_0 \longmapsto ?$$

$$\cap$$

$$D^b\,\mathrm{Coh}(\mathbb{C}^n) \simeq \mathcal{F}\left((\mathbb{C}^*)^n, \check{y}_1 + \dots + \check{y}_n\right)$$

n = 1





$$\mathrm{ext}^*(\mathcal{O}_0, \mathcal{O}_0) = \Lambda^*(b) \qquad \hom^*(L_0, L_0) = \Lambda^*(b)$$

Mirror symmetry heuristic

$$\mathcal{O}_0 \longmapsto \ ?$$
$$\cap$$
$$D^b Coh(\mathbb{C}^n) \simeq \mathcal{F}((\mathbb{C}^*)^n, \check{y}_1 + \dots + \check{y}_n)$$

$n > 1$

Use Künneth

Need more flexible definition of $\mathcal{F}$

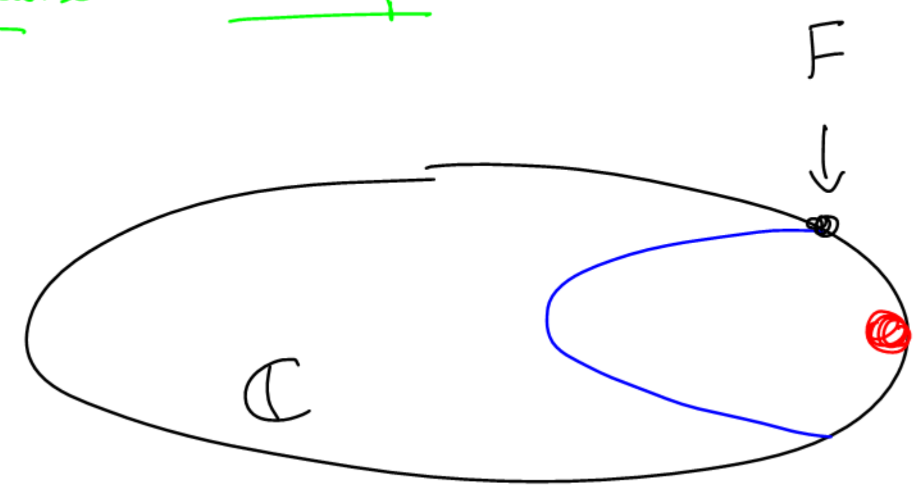
$\mathcal{F}(\check{w})$ à la Ganatra–Pardon–Shende, Sylvan, ...

Objects Exact Lagrangians $L \subset \mathbb{C}^n$ with cylindrical ends such that $\partial_\infty L$ is disjoint from $\check{w}^{-1}(+\infty)$ in $\partial_\infty \mathbb{C}^n$

L may be fibrewise non-compact

eg $L = \cup F$

possibly non-compact Lagrangian in Milnor fibre



$\mathcal{F}(\check{w})$ à la Ganatra-Pardon-Shende, Sylvan, ...

Morphisms $\hom^*(K, L) = CW^*(K, L)$

wrapped in $\partial_\infty \mathbb{C}^n$
stopped at $\check{w}^{-1}(+\infty)$

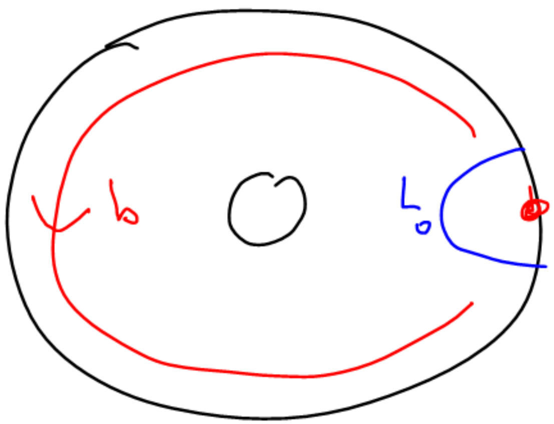
Remark The two definitions are equivalent after $\Pi\,\mathrm{Tw}$

→ both are split-generated by Lefschetz thimbles

Theorem (GPS) A Künneth formula holds

Upshot

$$\begin{array}{ccc} D^b\mathrm{Coh}(\mathbb{C}^n) & \simeq & \mathcal{F}\big((\mathbb{C}^*)^n, \check{y}_1+\dots+\check{y}_n\big) \\ \rotatebox{90}{$\ni$} & & \wr \\ \mathcal{O}_0 & & \mathcal{F}\big((\mathbb{C}^*, \check{y})^{\times n}\big) \\ & \longleftrightarrow & \rotatebox{90}{$\ni$} \\ & & L_0^{\times n} \end{array}$$



$$\hom^*(L_0^{\times n}, L_0^{\times n}) = \Lambda^*(b_1, \dots, b_n)$$

Remark $|b_i| = 1$ with respect to grading $\left(\frac{d\check{y}_1}{\check{y}_1} \wedge \dots \wedge \frac{d\check{y}_n}{\check{y}_n}\right)^2$

Passing to cover

$$((\mathbb{C}^*)^n, \check{w}(\check{x}_1, \dots, \check{x}_n)) \longrightarrow ((\mathbb{C}^*)^n, \check{y}_1 + \dots + \check{y}_n)$$

$$\check{w} = \sum_{i=1}^{n} \prod_{j=1}^{n} \check{x}_j^{a_{ji}} \qquad \check{y}_i = \prod_{j=1}^{n} \check{x}_j^{a_{ji}}$$

Deck group $\quad \{(\check{x}_j \longmapsto e^{i\theta_j}\check{x}_j) : A^T \underline{\theta} \in 2\pi\mathbb{Z}^n\}$

$$\|$$

G = character group $\hat{\Gamma}_w \quad \rightsquigarrow \quad G/\langle x_w \rangle = \overline{G} \quad$ finite

Γ_w = symmetries of w

$$(t_1, \dots, t_{n+1}) \longmapsto t_{n+1}$$

Passing to cover

$$\left((\mathbb{C}^*)^n,\ \check{w}(\check{x}_1,\dots,\check{x}_n)\right) \longrightarrow \left((\mathbb{C}^*)^n,\ \check{y}_1+\dots+\check{y}_n\right)$$

$$|\bar{G}| = \det A \quad \rightsquigarrow \quad L_0^{\times n}$$

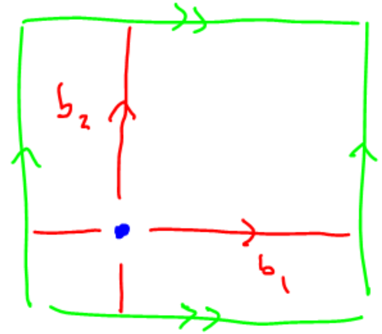
lifts

- Lifts form a torsor for $\bar{G}$
- Let $\mathbb{L} = \bigoplus$ lifts
- $\hom^*(\mathbb{L},\mathbb{L})$ = quiver algebra on $T^n \subset (\mathbb{C}^*)^n$

Passing to cover

Example $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ $\qquad \check{w} = \check{x}_1^2 + \check{x}_1 \check{x}_2^2$

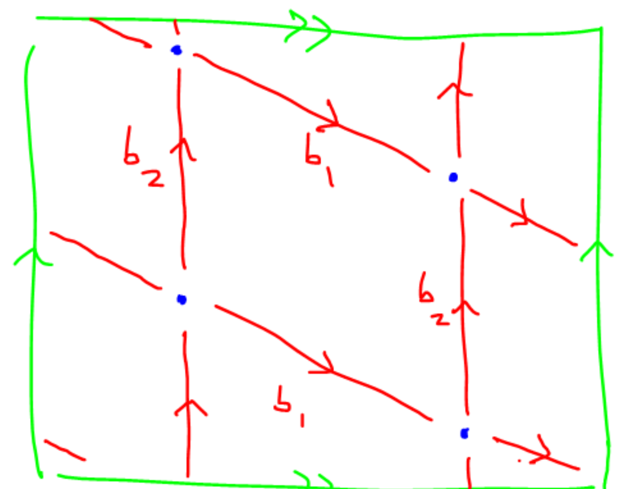
$\hom^*(L_0^{\times 2}, L_0^{\times 2}) =$



b_i graded-commute

$\downarrow (A^T)^{-1} = \frac{1}{4}\begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}$

$\hom^*(\mathbb{L}, \mathbb{L}) =$



b_i graded-commute

Partially compactify $\left((\mathbb{C}^*)^n, \check{w}\right) \rightsquigarrow \left(\mathbb{C}^n, \check{w}\right)$

$$\begin{array}{ccc} (\mathbb{C}^*)^n & & \mathbb{C}^n \\ \cup & & \cup \\ \mathbb{L} & \longmapsto & \mathbb{L} \end{array}$$

$hom^*_{\mathbb{C}^n}(\mathbb{L}, \mathbb{L})$ should be A_∞-deformation of $hom^*_{(\mathbb{C}^*)^n}(\mathbb{L}, \mathbb{L})$

No room for deformations at cohomology level

So $Hom^*_{\mathbb{C}^n}(\mathbb{L}, \mathbb{L})$ should be the same quiver algebra

coincides with $Hom^*_{mf}(\mathcal{O}, \mathcal{O})$

where $\mathcal{O}$ = sum of Γ_w-twists of $\mathcal{O}_0$

Gradings

- As ungraded Lagrangians the lifts of $L_0^{\times n}$ form a torsor for $\bar{G} = G/\langle x_w \rangle$

- As graded Lagrangians in $(\mathbb{C}^*)^n$ they form a torsor for $\mathbb{Z} \oplus \bar{G}$

with grading $\left(\frac{d\check{x}_1}{\check{x}_1} \wedge \dots \wedge \frac{d\check{x}_n}{\check{x}_n} \right)^2$

homological

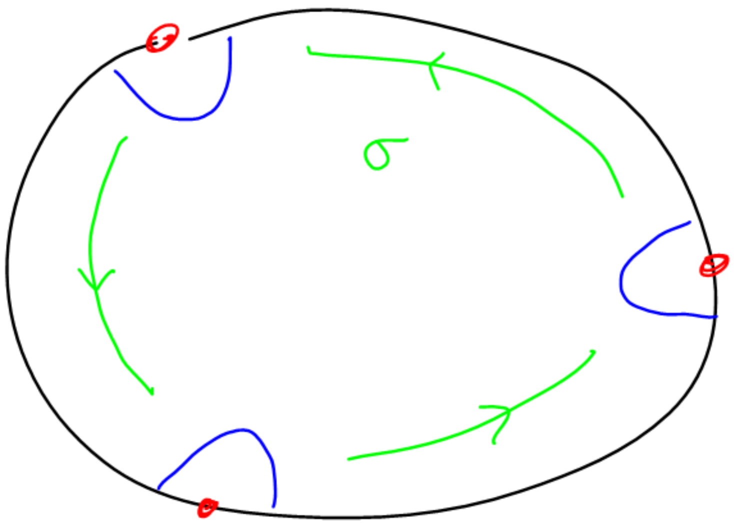
internal

- As graded Lagrangians in $\mathbb{C}^n$ this becomes

$\tilde{G} = \mathbb{Z} \oplus G / (2, x_w)$

with grading $(d\check{x}_1 \wedge \dots \wedge d\check{x}_n)^2$

Picture in $n=1$ $\qquad \check{w} = \check{x}_1^3 \qquad w = x_1^3$



σ^3 acts as $[-2]$

$$\Gamma_w = \{(t,t^3) : t \in \mathbb{C}^*\} \cong \mathbb{C}^*$$

project to t

$$G = \hat{\Gamma}_w = \langle \sigma \rangle \cong \mathbb{Z}$$

$$x_w = \sigma^3$$

$$\bar{G} = G / \langle x_w \rangle = \langle \sigma \rangle / \langle \sigma^3 \rangle \cong \mathbb{Z}/3$$

$$\tilde{G} = \mathbb{Z} \oplus G / (2, x_w) = \mathbb{Z} \oplus \langle \sigma \rangle / (2, \sigma^3)$$

Gradings As graded Lagrangians in $\mathbb{C}^n$ the lifts of $L_0^{\times n}$ form a torsor for $\widetilde{G} = \mathbb{Z} \oplus G \big/ (2, \chi_\omega)$

Crucial consequence

- $\hom^*(\mathbb{L}, \mathbb{L})$ is a $\widetilde{G}$-graded algebra
- Yoneda modules $\hom^*(\mathbb{L}, -)$ are naturally relatively $\widetilde{G}$-graded
- choosing a base point for the torsor $\{$graded lifts$\}$ makes this relative grading absolute

this choice is the only ambiguity in our mirror functor

Recap

- Constructed $L_0^{\times n} \subset (\mathbb{C}^*)^n$ mirror to $\mathcal{O}$.
  using Künneth
- Passed to cover to obtain
  $\mathbb{L} = \bigoplus$ lifts $\circlearrowleft$ deck group $\underline{}$ G
- Singling out one graded lift of $L_0^{\times n}$ makes
  $\hom^*(\mathbb{L}, -)$ $\tilde{G}$-graded

$\underline{mf(\mathbb{C}^n, \Gamma_w, w)}$ revisited

$$\cdots \longrightarrow E^0 \xrightarrow{\delta} E^1 \xrightarrow{\delta} E^2 \longrightarrow \cdots$$

- E^i Γ_w-equivariant vector bundle on $\mathbb{C}^n$
- $E^{i+2} = E^i \otimes \chi_w$
- $\delta^2 = \mathrm{id}_{E^i} \otimes w$

$\chi_w =$ character $(t_1, \dots, t_{n+1}) \longmapsto t_{n+1}$

$\underline{mf(\mathbb{C}^n, \Gamma_w, w)}$

$$\cdots \longrightarrow E^0 \xrightarrow{\delta} E^1 \xrightarrow{\delta} E^2 \longrightarrow \cdots$$

- E^i G-graded vector bundle on $\mathbb{C}^n$

  $G = \hat{\Gamma}_w$ character group

- $E^{i+2} = E^i(-\chi_w)$

- $\delta^2 = w \cdot id_{E^i}$

χ_w = character $(t_1, \ldots, t_{n+1}) \longmapsto t_{n+1}$

$\underline{mf(\mathbb{C}^n, \Gamma_w, w)}$

$$\cdots \longrightarrow E^0 \xrightarrow{\delta} E^1 \xrightarrow{\delta} E^2 \longrightarrow \cdots$$

Total object $E = \bigoplus_{i \in \mathbb{Z}} E^i$ is graded

by $\tilde{G} = \mathbb{Z} \oplus G$ $(2, \chi_w)$

homological

internal

Where have we got to?

- Have A_∞ Yoneda functor

$$\hom^*(\mathbb{L}, -) : \mathcal{F}(\check{w}) \longrightarrow \tilde{G}\text{-graded chain cxes}$$

- Identified $mf(\mathbb{C}^n, \Gamma_w, w)$ with $\tilde{G}$-graded "complexes" satisfying $\delta^2 = w$

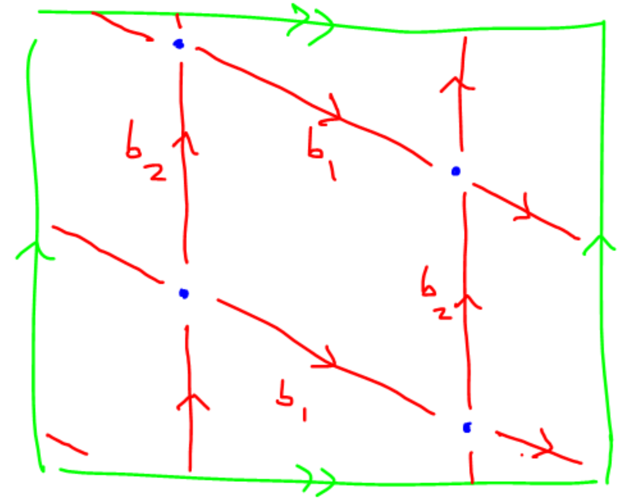
To achieve $\delta^2 = w$ equip $\mathbb{L}$ with a weak bounding cochain

$$\underline{b} \in \hom^1(\mathbb{L}, \mathbb{L}) \otimes_{\mathbb{C}} \mathbb{C}[x_1, \dots, x_n]$$

satisfying $\displaystyle\sum_{k=1}^{\infty} \mu^k(\underline{b}, \dots, \underline{b}) = w \cdot 1_{\mathbb{L}}$

sum finite by gradings

Recall

$\mathrm{Hom}^*(\mathbb{L}, \mathbb{L}) =$ 

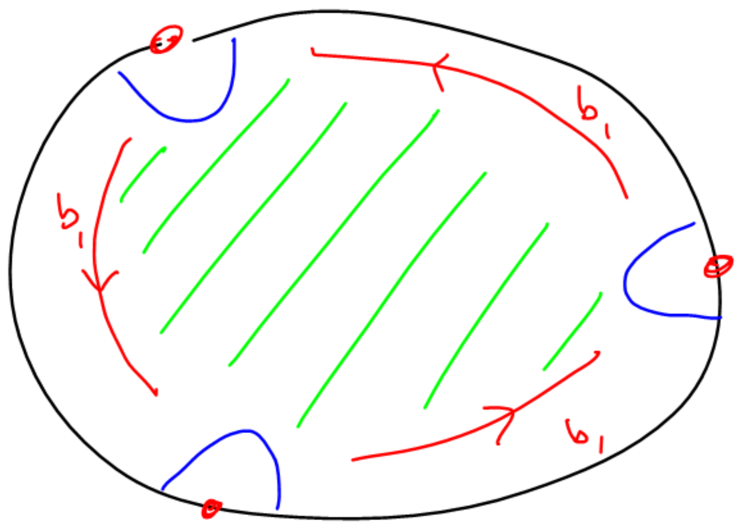
b_i graded-commute

Define $\underline{b} = x_1 b_1 + \ldots + x_n b_n \in \hom^1(\mathbb{L}, \mathbb{L}) \otimes_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_n]$

Proposition For $\tilde{G}$-grading reasons this is a wbc satisfying $\sum_{k=1}^{\infty} \mu^k(\underline{b}, \ldots, \underline{b}) = P \cdot 1_{\mathbb{L}}$ where P is a sum of a subset of the monomials in W

"Obvious" conjecture All monomials appear

Back to $n=1$ example



Green disc gives

$$\mu^3(b_1, b_1, b_1) = 1_L$$

all other $\mu^k(b_1, \dots, b_1)$ vanish

So $$\sum_{k=1}^{\infty} \mu^k(\underline{b}_1, \dots, \underline{b}_1) = x_1^3 \cdot 1_L$$

Finishing off

Obtain $\hom^*((\mathbb{L}, \underline{b}), -) : \mathcal{F}(\check{w}) \to mf(\mathbb{C}^n, \Gamma_w, w)$

localised mirror functor (Cho-Hong-Lau)

Spectral sequence shows $\mathbb{L} \xmapsto{\sim} \mathbb{C}$

(cf S. '19 for monotone tori)

known to split-generate (Dyckerhoff, Orlov, Seidel, ...)

Then $\mathbb{L}$ split-generates $\mathcal{F}(\check{w})$ by an automatic generation argument

Ganatra, Seidel, Tabuada

$\blacksquare$

What remains to check

- $\mathrm{Hom}^*(\mathbb{L}, \mathbb{L})$ is expected quiver algebra

  would follow if Hom in $\mathbb{C}^n$ were a deformation of Hom in $(\mathbb{C}^*)^n$

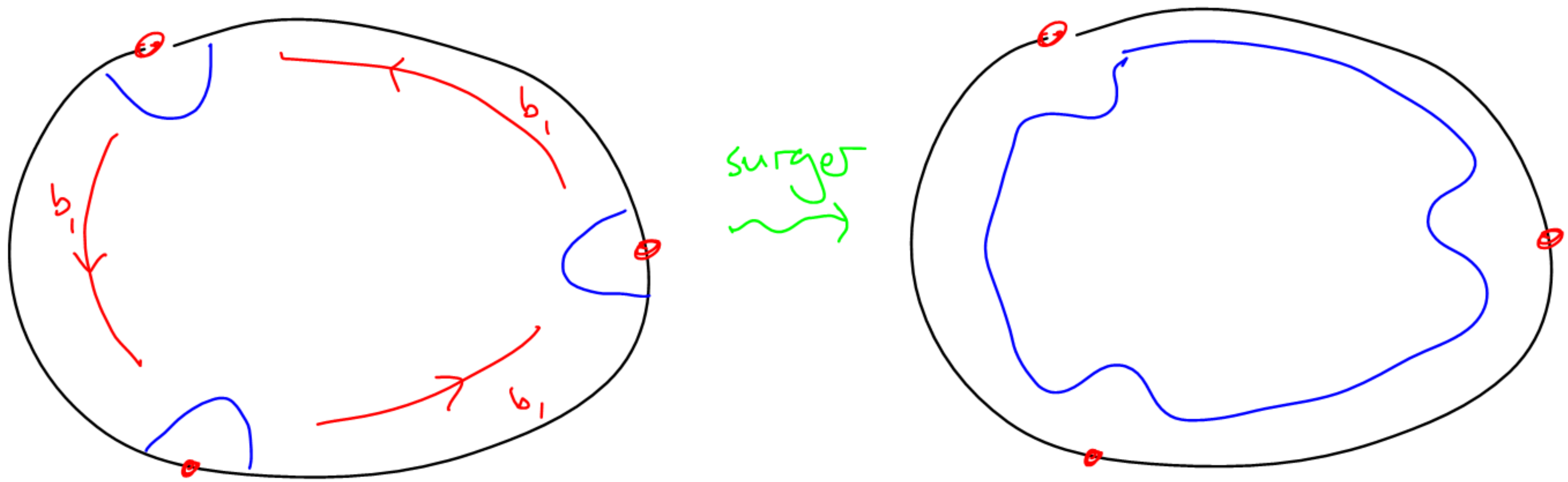
- The curvature/potential $P = \sum_{k=1}^{\infty} \mu^k(\underline{b}, \ldots, \underline{b})$ contains all expected monomials

  would follow from HH_* computation if we separately knew $\mathbb{L}$ split-generated

Final thought: what is $(\mathbb{L}, \underline{b})$?

- Incorporating a wbc is like taking a self mapping cone
- Mapping cones correspond to surgeries
- Surgering $\mathbb{L}$ at $\underline{b}$ gives a torus at infinity mirror to the origin

$n=1$



Thanks for listening!